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The relationship between tensile and bending properties of non-linear composite materials

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The effect of a Weibull distribution of flaws on the strength on non-linear composite materials is considered. The tensile strength applying in bending is compared with that in direct tension. This leads to an estimate of the ratio modulus of rupture to ultimate tensile strength for composite materials.

1. Introduction

In recent years considerable research effort has been devoted to the development of composite materials based on a brittle matrix, notably cement or gypsum plaster reinforced with glass or other fibres. These new materials are usually non-linear, although some, for example grc, become brittle under some conditions of storage [1]. The application of these new materials requires detailed knowledge of their response to stress, and the interrelationship between the response to stresses differently applied. The immediate need is to define the relationship between behaviour in tension and in bending.

For linear materials free from flaws, the tensile strength measured directly and that calculated from bending should be equal. In practice this is usually not the case and the material shows a distribution of breaking strengths; an adjustment has then to be made for the effect of the non-uniform application of the stress in bending. Weibull [2] has proposed a statistical method to determine the strength of a brittle material. The method allows for specimen size, a scatter of failure strengths (i.e. flaws) and a distribution of applied stresses.

Where the material is non-linear, and shows a distribution of flaws, the situation is more complicated. Previous work has defined the response in bending given a knowledge of the actual tensile stress–strain curve [3], or that predicted from fibre and matrix properties [4], but neglects the effect of a distribution of materials strength. Later work [5] has shown how the tensile and compressive stress–strain curves can be derived from the results of a bending test in which the strains on both faces are recorded as the load is applied. The tensile stress–strain curve that results defines the tensile properties operating in bending, for the size and shape of the test piece involved, and does not necessarily apply in direct tension. Indeed the curves calculated from bending and those measured directly show significant differences [5].

The present paper extends the analysis of the effect of a distribution of flaws on the bending strength of linear materials, to non-linear materials. In particular it considers fibrous composites, assuming a Weibull distribution of failure strengths. This allows the tensile strength applying in bending to be compared with that in direct tension.

The results of the two analyses, namely, the effect of a distribution of failure strengths, and the relationship between bending and tensile response resulting from the non-linearity of the tensile stress–strain curve, taken together, lead to an estimate of the ratio of modulus of rupture to tensile strength for non-linear materials.

2. Theoretical considerations

The probability of failure, P, of a material that has a Weibull distribution of flaws throughout its volume, V, is given [1] by the expression:

$$P = 1 - \exp\left[-\int \phi(\sigma)\, \mathrm{d}V\right], \quad (1)$$

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where $\phi(\sigma)$ is a function expressing the strength properties of the material. Weibull proposed using the function

$$\phi(\sigma) = \left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m, \quad (2)$$

where σ is the applied stress; m is a constant related to the flaw size distribution and is known as the Weibull modulus; σ_u is the stress below which no failures occur (usually assumed to be zero); and σ_0 is a normalizing factor.

Using the function 2, Equation 1 becomes

$$P = 1 - \exp\left[-\int_V \left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m \mathrm{d}V\right]. \quad (3)$$

In the following analyses, it is assumed that $\sigma_u = 0$.

2.1. Linear materials

In the case of a uniform linear material free from flaws, the strength in bending or 'modulus of rupture' (MOR), and the tensile strength are identical, and there is no 'size effect'. In practice this is rarely the case: most materials show a scatter of failure strengths and the measured tensile strength depends on the size of the specimen tested.

Assuming a Weibull distribution of materials strength, Equation 3 can be used to calculate the probability of failure when a uniform tensile stress is applied over a volume V_t:

$$P_t = 1 - \exp\left[-V_t\left(\frac{\sigma_t}{\sigma_0}\right)^m\right]. \quad (4)$$

When a rectangular beam of the same material, of volume V_b, is subjected to equal four-point loading, the stress varies linearly with distance from the neutral axis, and (see, for example, [6])

$$P_b = 1 - \exp\left[-V_b\left(\frac{\sigma_b}{\sigma_0}\right)^m \frac{(m+3)}{6(m+1)^2}\right], \quad (5)$$

where σ_b is the maximum tensile stress developed on the outer tensile face of the beam.

From Equations 4 and 5 it follows that, for equal probabilities of failure, the ratio of the strength in flexure, σ_b, to the strength in direct tension σ_t, is

$$\frac{\sigma_b}{\sigma_t} = \left[\frac{V_t}{V_b}\frac{6(m+1)^2}{(m+3)}\right]^{1/m}. \quad (6)$$

Broutman and Krock [7] relate the Weibull modulus m to the coefficient of variation (C of V) of strength. For the 10% C of V commonly found for glass reinforced cement (grc) produced and tested at BRE, m is approximately 10. From Equation 6 it follows that for samples of equal width and thickness, tested in equal four-point bending (total span 135 mm) and in tension (approx 65 mm gauge length)

$$\frac{\sigma_b}{\sigma_t} \sim 1.4.$$

Alternatively, the Weibull modulus can be calculated from tensile strength data – it is the slope of the plot of ln ln $[1/(1-P)]$ against ln σ_t.

2.2. Non-linear materials

If the material is non-linear, the situation is more complicated. Firstly, the stress across the beam in bending does not change linearly with distance from the neutral axis, and therefore Equation 5 above does not apply. However, provided that the stress distribution in the beam is known, the probability of failure can be calculated from Equation 3, and compared with the probability of failure under uniaxially applied tension to give the ratio of tensile strength applying in bending, to the strength in uniaxial tension.

Secondly the shape of the tensile stress–strain curve affects the maximum load that can be supported in bending. This happens even if the material is uniform in properties and does not exhibit a scatter of failure strengths. The result is that the apparent modulus of rupture is greater than the tensile strength [4]. This effect can also lead to an increase in the strain on the tensile face when the maximum load is reached, beyond the failure strain in direct tension [8].

In the analysis that follows it is assumed that the distribution function, Equation 3, applies and $\sigma_u = 0$. Obviously the analysis could be performed using any other suitable distribution function.

3. Procedure

This procedure is as follows:

(i) The tensile stress–strain curve is used to calculate the probability of failure of a beam in, for example, four-point bending, assuming a volumetric flaw distribution, and using Equation 3 above.

Because of the non-linearity of the stress–strain response in tension, the probability of failure in bending has to be calculated numerically. This requires that the distribution of stress both along the length of the beam and through its thickness, is known.

In the central portion of the beam, where the bending moment is constant, the stress distribution through the thickness of the beam when the stress on the tensile face is σ_b, is simply calculated from the balance of forces condition. This distribution is constant along the length of the beam between the central supports.

Over the outer lengths of the beam, the bending moment varies linearly with distance. The procedure then, is to calculate the bending moment as a function of increasing tensile strain and hence to relate the distance along the beam to the stress on the tensile face. The stress distribution through and along the beam is thus defined, and the full integration implied in Equation 3 can be performed numerically.

The stress that would be needed to produce equal probability of failure in direct tension is then calculated from Equation 4. This gives the ratio σ_b/σ_t, the 'size effect' correction.

(ii) The tensile stress–strain curve is used to calculate the apparent maximum tensile stress in the beam at failure (modulus of rupture, MOR, calculated from elastic bending theory). The method of calculation (see, for example, [9]) is well estabished. The tensile stress–strain curve used should be that applying in bending (tensile strength σ_b). The ratio of modulus of rupture to tensile strength, MOR/σ_b, is that arising from non-linearity of the tensile curve (the 'shape factor').

(iii) The product $(\sigma_b/\sigma_t) \times (\text{MOR}/\sigma_b)$ gives the expected ratio MOR/σ_t that allows for both the size effect and the shape factor. This is commonly described as the MOR/UTS (ultimate tensile strength) ratio.

4. Application and discussion

4.1. Comparison of tensile strengths applying in bending and in uniaxial tension

Previous work [5, 9] has shown that the expected bending response calculated from the measured tensile stress–strain curve commonly falls short of that observed experimentally. In particular the strain on the tensile face at maximum bending load is often very much higher than the failure strain in direct tension. Part of this increase can be explained if there is a stress capacity after failure in tension [8, 9], but a discrepancy still remains. This is confirmed by recent work [5] in which tensile stress–strain curves deduced from bending data, were compared with those measured directly. For brittle and ductile materials alike (an asbestos cement and a ductile grc), both the stresses and strain at failure deduced from bending were higher than those measured.

A possible reason for the discrepancy could be the effect of a distribution of failure strengths. Assuming a Weibull modulus of 10, and the measured tensile stress–strain curves shown in Figs. 1 and 2, the calculated ratios σ_b/σ_t for the specimen dimensions used (tensile gauge length 65 mm; four-point bending with inner and outer spans of 134 mm and 289 mm, respectively) are 1.12 and 1.05 for the asbestos cement and the grc, respectively. The specimens were 50 mm wide and approximately 9 mm thick. The cal-calculated value of σ_b is 24.5 MN m^{-2} for asbestos cement and compares closely with the value of 24 MN m^{-2} deduced from bending data. In the case of the grc the comparison is not quite so close – the value of σ_b calculated from the tensile curve is 13.1 MN m^{-2}, compared with 14.5 MN m^{-2} deduced from the bending data.

While the calculated ratios σ_b/σ_t are low in both cases, they nevertheless have a significant effect, since even a small increase in failure stress implies a large increase in strain and correspondingly large effect on the modulus of rupture.

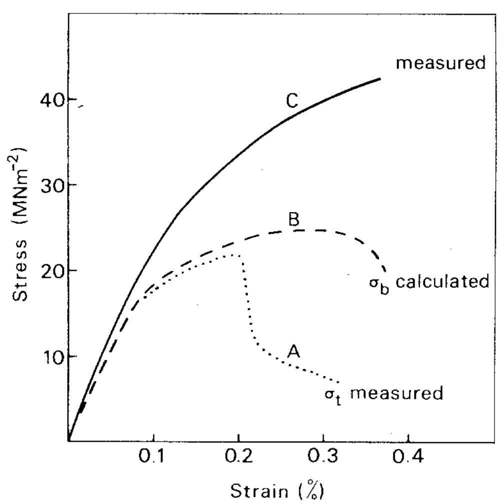

Figure 1 Tensile stress–strain curves (A and B) for an asbestos cement. Curve A was measured in uniaxial tension; curve B was calculated from bending data. Curve C is the measured apparent bending stress–strain curve corresponding to curve B.

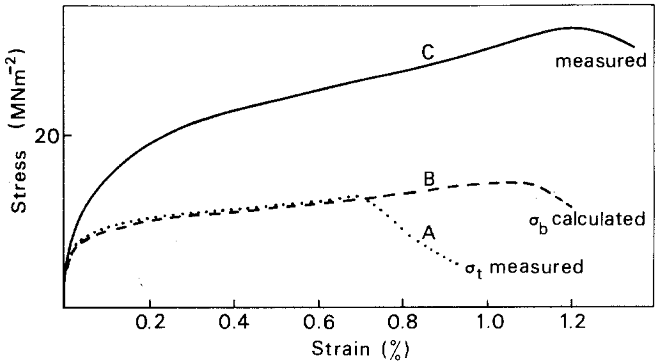

Figure 2 As Fig. 1 for a ductile grc.

Furthermore in both cases, the ratio of the volumes of the specimens tested in tension and in bending is low (0.22). For a volume ratio of 1, the ratios σ_b/σ_t predicted from the tensile stress–strain curves would be 1.30 and 1.22, respectively. These ratios are somewhat lower than that (1.46) which would apply to linear materials tested under the same conditions, a result of the 'yielding' of the composite after the matrix begins to crack. For the equal four-point bending rig used for the main test programme at BRE (total span 135 mm), the calculated σ_b/σ_t ratio for the ductile grc is 1.13, leading to a predicted MOR some 10% or more higher than that predicted if the larger rig is used. This is consistent with the results reported by Singh *et al.* [10] for the effect of span–depth on the MOR of grc. The data of Singh *et al.* were based on tests of the same grc board described above (and in Fig. 2).

4.2. The MOR/UTS ratio of grc

It has been reported previously [1] that the ratio of modulus of rupture (MOR) to ultimate tensile strength (UTS) of grc kept in both dry and wet environments remains approximately constant at about 2.5 : 1 for all ages beyond the early stage of cure. At first sight this is surprising particularly for prolonged storage in wet environments since grc loses its ductility under these conditions and becomes essentially a brittle material. The MOR/UTS ratio might then be expected to decrease with time of storage as the tensile stress–strain curve approaches linearity.

In some cases, particularly for samples subjected to natural weathering, even higher MOR/UTS ratios have been observed [11], and there is a tendency for these values to increase with age and with decreasing glass content. While part of this increase can be attributed to the increasing difficulties in measuring the tensile strength as the material becomes more brittle, it is important to examine the possibility of there being an alternative explanation.

In previous theoretical treatments (see, for example, [4]) the effect of a distribution of materials strength has been neglected and the MOR/UTS ratio calculated is equivalent to the MOR/σ_b ratio of this present work. Experimentally the MOR/UTS ratio measured is equivalent to MOR/σ_t. It would be useful then to consider even approximately, how the MOR/UTS ratios might be expected to vary with age if both 'shape' (MOR/σ_b) and 'size' (σ_b/σ_t) factors were included.

Curve A of Fig. 3 shows a representative tensile stress–strain curve for grc after an initial damp cure of 28 days. The measured curve has been extended to higher strains (dotted line) since there is evidence that the tensile stress–strain curve applying in bending is extended to higher strains. Curve B is the stress–strain relationship in bending calculated from Curve A. If ageing were simply to reduce the strain at failure along Curve A, the appropriate MOR/σ_b can be calculated by comparing the apparent bending stress and the corresponding tensile stress at the same strain, e.g. points B′ and A′. The MOR/σ_b ratio increases slightly and then decreases as expected as the strain decreases (Fig. 4); the increase results from the inflection in the tensile stress–strain curve.

The effect of size (σ_b/σ_t) can also be calculated, and is also shown in Fig. 4. This is opposite in direction to the effect of shape (MOR/σ_b), the ratio decreasing and then increasing as the strain is reduced.

In estimating the MOR/UTS ratio, point B′ (Fig. 3) should be compared not with point A′ (i.e. the tensile stress applying in bending), but with Point A″ ($= A'/(\sigma_b/\sigma_t)$), the stress applying in direct tension. The resulting relationship MOR/σ_t is seen (Fig. 4) to remain essentially constant

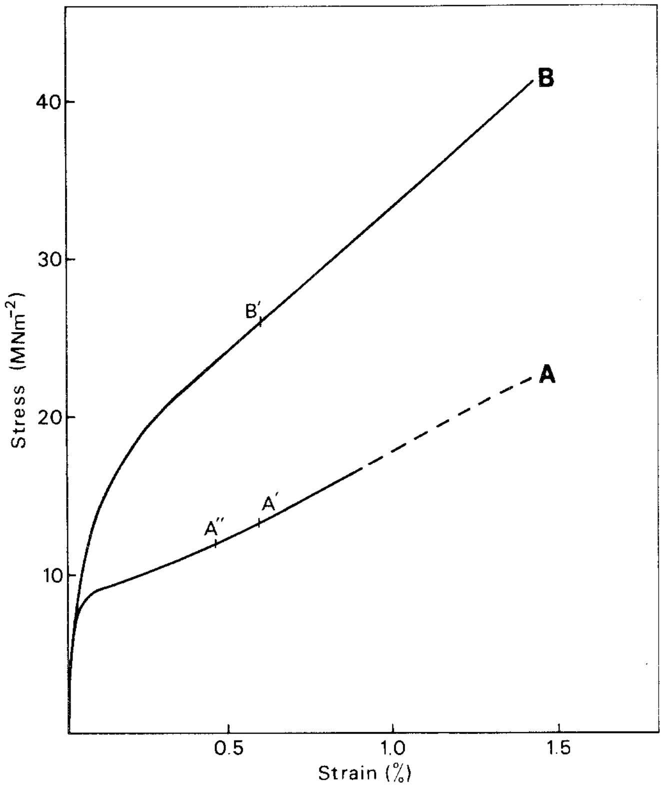

Figure 3 Typical stress–strain curve in tension for grc (curve A) extended to represent stress–strain curve applying in bending; and bending curve calculated from it (B).

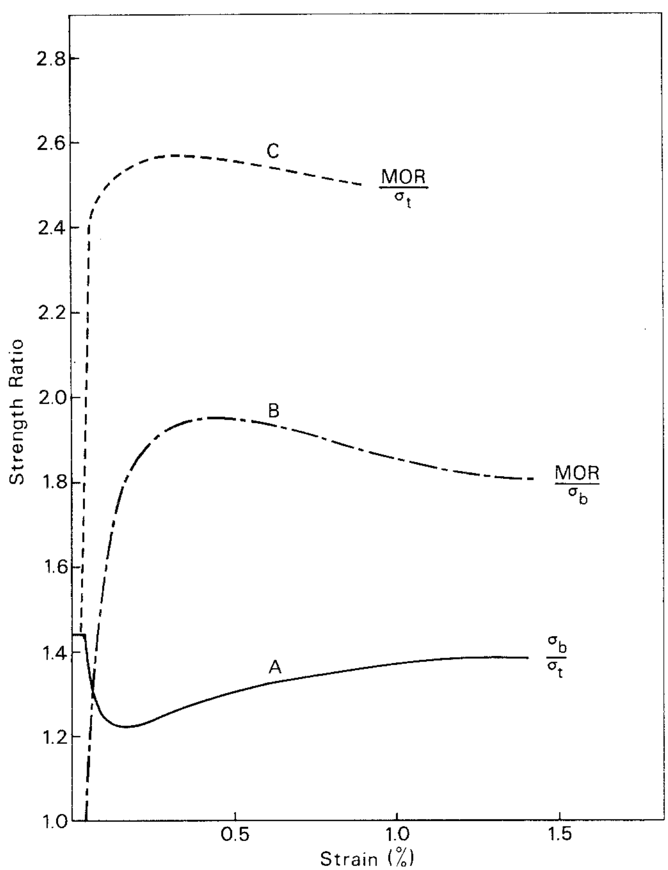

Figure 4 Strength ratios σ_b/σ_t (curve A), MOR/σ_b (curve B), and MOR/σ_t (curve C) as a function of (tensile) strain. A and B refer to strain in bending and curve C to strain in direct tension.

over most of the strain (i.e. assumed age) range above the elastic limit.

In this illustration, a Weibull modulus of 10 has been used, and there is some guesswork in the extension of the directly measured tensile stress–strain curve. It has been assumed that equal volumes are stressed in the two modes (direct tension and bending). Nevertheless the calculations suggest that a near constant ratio of MOR/σ_t would be expected as grc ages if the strain/age relationship holds: and that the ratio would be in the region of the MOR/UTS ratios observed experimentally. A similar result obtains if it is assumed that ageing reduces the effective volume fibre fraction. A more reliable test of the validity of the method would require tensile stress–strain curves as a function of age, and in particular those applying in bending, but sufficient data are not presently available for analysis.

Nevertheless the analysis does suggest an explanation of the otherwise unexpected results.

5. Conclusions

It appears that the Weibull theory can be applied usefully to non-linear materials such as fibre-reinforced cements, to predict the relationship between the tensile strength applying in bending and that applying in direct tension. There is experimental support that the predicted ratios are realistic. In particular the analysis leads to a predicted size effect for a ductile grc that is observed experimentally.

In predicting the modulus of rupture from measured tensile stress–strain curves the effect of a distribution of materials strength is normally neglected. When the size effect is included, the predicted MOR more closely accords with that measured.

The inclusion of a factor for 'size' as well as a factor for the shape of the tensile stress–strain curve offers an explanation for the observed near constant MOR/UTS ratio for grc as it ages.

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